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# The de Rham complex on unbounded domains with Sobolev space topology

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**Abstract**

This paper studies the operator $dd^* + d^*d$ acting on $q$-forms on an unbounded domain with smooth boundary, where $d$ is the exterior derivative and $d^*$ is the adjoint of $d$ calculated using the Sobolev space topology. The domain of $d^*$ is determined and an expression for $d^*$ is obtained. The operator $dd^* + d^*d$ gives rise to a boundary value problem. Global regularity is obtained using weighted norms and global existence is obtained by using the theory of compact operators.


*Keywords:* de Rham complex; Sobolev topology; Elliptic boundary value problems; Pseudo-differential boundary problems

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{N+1}$ having smooth boundary. Consider the exterior derivative operator $d$ acting on $q$-forms with coefficients in $L^2(\Omega)$:

$$L^2_q(\Omega) \xrightarrow{d} L^2_{q+1}(\Omega) \xrightarrow{d} \cdots.$$

Let $d^*$ be the adjoint of $d$ in the $L^2$ topology. In [3], the elliptic operator $dd^* + d^*d$ is studied.

The equation $(dd^* + d^*d)\phi = \alpha$ makes sense only if $\phi \in \operatorname{dom} d^*$ and $d\phi \in \operatorname{dom} d^*$, which are actually boundary conditions. Thus, the author studies the boundary value problem

$$\begin{cases} (dd^* + d^*d)\phi = \alpha \quad \text{in } \Omega, \\ \phi \in \operatorname{dom} d^*, \\ d\phi \in \operatorname{dom} d^*. \end{cases} \tag{1}$$

The author gives necessary conditions for $(dd^* + d^*d)\phi = \alpha$ to be solvable and regularity statements are also given.

In their paper *Hodge Theory in the Sobolev Topology for the de Rham Complex*, Fontana, Krantz and Peloso study the boundary value problem (1) on smoothly bounded domains and the upper half space using the Sobolev topology

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to calculate the adjoint operator. As a result, the domain of the adjoint operator is different from the classical case, and a new boundary value problem arises. They give a complete existence and regularity theory for smoothly bounded domains and for the upper half space.

In this paper, we will prove corresponding results on an unbounded domain $\Omega$ with smooth boundary. As in [2], we see that the Hodge operator $\square = dd^* + d^*d$ is an elliptic pseudo-differential operator. We will establish a condition for when $\phi$ and $d\phi$ are in dom $d^*$, and we will give an expression for $d^*$ (calculated in the $W^1$ topology). A global regularity theory for solutions $\phi$ to the boundary value problem is obtained by using weighted norms and, from this result and some functional analysis, we establish a global existence theory of solutions to the boundary value problem.

## 2. The domain of $d^*$

In order to study the boundary value problem (1), we need to first determine $(\operatorname{dom} d^*) \cap \Lambda^q(\overline{\Omega})$, and to give an expression for $d^*$.

Let $x' = (x_0, x_1, \dots, x_{N-1})$ and $f \in C^\infty$. Let $\Omega = \{(x', x_N) \in \mathbb{R}^{N+1} \mid x_N > f(x')\}$.

**Proposition 1.** *The $(q+1)$-form $\psi$ lies in $(\operatorname{dom} d^*) \cap \Lambda^{q+1}(\overline{\Omega})$ if and only if*

$$\sum_{|I|=q} \sum_{\substack{|K|=q+1\\ j=0\dots N}} \left(\epsilon^K_{jI} \frac{\partial \psi_K}{\partial n} \frac{\partial \rho}{\partial x_j}\right) dx^I = \nabla_{N_1} \psi \lfloor N_1 = 0 \quad \text{on } b\Omega,$$

*where $N_1$ is the normal vector field on $\Omega$ and $\lfloor$ is the contraction operator.*

**Proof.** For the unbounded domain $\Omega$ with smooth boundary, we will assume that the coordinates are normalized so that $-1 \leqslant f(x') \leqslant 1$ whenever $-1 \leqslant x_j \leqslant 1$, $j = 0, \dots, N-1$.

Recall that, by definition, $\psi \in \operatorname{dom} d^*$ if and only if there is a constant $c_\psi$ such that for all $\phi \in \Lambda^q$, we have

$$\left|\langle d\phi, \psi\rangle_1\right| \leqslant c_\psi \|\phi\|_1.$$

Notice that

$$\langle d\phi, \psi\rangle_1 = \langle \phi, d'\psi\rangle_1 + \sum_{\substack{|I|=q\\ k=0\dots N}} \int_{b\Omega} D_k\phi_I \left(\overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} D_k\psi_K \frac{\partial \rho}{\partial x_j}}\right) + \sum_{|I|=q} \int_{b\Omega} \phi_I \left(\overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} \psi_K \frac{\partial \rho}{\partial x_j}}\right),$$

and that the first and last terms are bounded by $c_\psi \|\phi\|_1$. Thus, we consider the remaining term

$$\sum_{k=0}^N \int_{b\Omega} D_k\phi_I \left[\overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} D_k\psi_K \frac{\partial \rho}{\partial x_j}}\right].$$

We write

$$D_k = T_k + \frac{\partial \rho}{\partial x_k} \frac{\partial}{\partial n},$$

where $T_k$ is the tangential part of $D_k$, in a suitable neighborhood of the boundary. So

$$\begin{aligned}
&\sum_{k=0}^N \int_{b\Omega} D_k\phi_I \overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} D_k\psi_K \frac{\partial \rho}{\partial x_j}} \\
&\quad = \sum_{k=0}^N \int_{b\Omega} T_k\phi_I \overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} D_k\psi_K \frac{\partial \rho}{\partial x_j}} + \sum_{k=0}^N \int_{b\Omega} \frac{\partial \rho}{\partial x_k} \frac{\partial \phi_I}{\partial n} \overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} D_k\psi_K \frac{\partial \rho}{\partial x_j}} \\
&\quad = \sum_{k=0}^N \int_{b\Omega} \phi_I T_k^* \overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} D_k\psi_K \frac{\partial \rho}{\partial x_j}} + \int_{b\Omega} \frac{\partial \phi_I}{\partial n} \overline{\sum_{\substack{|K|=q+1\\ j=0\dots N}} \epsilon^K_{jI} \frac{\partial \psi_K}{\partial n} \frac{\partial \rho}{\partial x_j}}.
\end{aligned}$$

Now, by the trace theorem,

$$\left|\sum_{k=0}^{N}\int_{b\Omega}\phi_I T_k^*\left[\overline{\sum_{\substack{|K|=q+1\\ j=0\ldots N}}\epsilon_{jI}^K D_k\psi_K\frac{\partial\rho}{\partial x_j}}\right]\right|\leqslant c_\psi\|\phi\|_1.$$

Thus if

$$\left[\overline{\sum_{\substack{|K|=q+1\\ j=0\ldots N}}\epsilon_{jI}^K\frac{\partial\psi_K}{\partial n}\frac{\partial\rho}{\partial x_j}}\right]\equiv(\nabla_{N_1}\psi\lfloor N_1)_I=0\quad\text{on } b\Omega,$$

then $\psi\in\operatorname{dom} d^*$.

Conversely, suppose that $\psi\in\operatorname{dom} d^*\cap\Lambda_0^{q+1}(\overline{\Omega})$ and $\nabla_{N_1}\psi\lfloor N_1\neq 0$ on the boundary of $\Omega$. We seek a contradiction.

We may suppose (by multiplying by a constant and scaling) that $\psi_K$, $|K|=q+1$, are real and

$$\sum_{\substack{|K|=q+1\\ j=0\ldots N}}\epsilon_{jI}^K\frac{\partial\psi_K}{\partial n}\frac{\partial\rho}{\partial x_j}\geqslant 1\quad\text{when } |x_j|\leqslant 2,\ j=0,\ldots,N.$$

Now, set $\phi_{I\epsilon}(x',x_N)=(x_N-f(x')+\epsilon)^{3/4}\chi(x',x_N)$, where $\chi$ is a $C^\infty$ function such that $\chi\equiv 1$ on $\{|x_0|\leqslant 1/2\}\times\cdots\times\{|x_N|\leqslant 1/2\}$ and $\chi\equiv 0$ on $\{|x_0|>1\}\times\cdots\times\{|x_N|>1\}$.

We claim that there is a constant $C>0$, independent of $\epsilon$, such that

(i) $$\|\phi_{I\epsilon}\|_{W^1(\Omega)}\leqslant C,\quad 0<\epsilon\leqslant\epsilon_0,$$

and

(ii) $$\int_{\mathbb{R}^N}\frac{\partial\phi_{I\epsilon}}{\partial n}\left[\overline{-\sum_{\substack{|K|=q+1\\ j=0\ldots N}}\epsilon_{jI}^K\frac{\partial\psi_K}{\partial n}\frac{\partial\rho}{\partial x_j}}\right]\Bigg|_{x_N=f(x')}\sim\frac{1}{\epsilon^{1/4}}\quad\text{as }\epsilon\to 0.$$

After establishing (i) and (ii), we will have our contradiction: it is not possible to find a constant such that the mapping $\phi\to\langle d\phi,\psi\rangle_1$ is bounded on $W_{q+1}^1(\Omega)$; thus $\psi$ cannot be in $\operatorname{dom} d^*$.

To prove (ii), we notice first that

$$\begin{aligned}\frac{\partial\phi_{I\epsilon}}{\partial n}(x',x_N)\Big|_{x_N=f(x')}&=\sum_{k=0}^{N-1}\frac{\partial f}{\partial x_k}\left[-\frac{3}{4}\epsilon^{-1/4}\left(\frac{\partial f}{\partial x_k}\right)\chi\big(x',f(x')\big)+\epsilon^{3/4}\frac{\partial\chi}{\partial x_k}\right]\\&\quad-\frac{3}{4}\epsilon^{-1/4}\chi\big(x',f(x')\big)-\epsilon^{3/4}\frac{\partial\chi}{\partial x_N}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{\partial\phi_{I\epsilon}}{\partial n}\left[\overline{-\sum_{\substack{|K|=q+1\\ j=0\ldots N}}\epsilon_{jI}^K\frac{\partial\psi_K}{\partial n}\frac{\partial\rho}{\partial x_j}}\right]\Bigg|_{x_N=f(x')}\\&\quad=\int_{\mathbb{R}^N}\left[\sum_{k=0}^{N-1}\left(\frac{\partial f}{\partial x_k}\left[-\frac{3}{4}\epsilon^{-1/4}\left(\frac{\partial f}{\partial x_k}\right)\chi\big(x',f(x')\big)+\epsilon^{3/4}\frac{\partial\chi}{\partial x_k}\right]\right)\right.\\&\qquad\left.-\frac{3}{4}\epsilon^{-1/4}\chi\big(x',f(x')\big)-\epsilon^{3/4}\frac{\partial\chi}{\partial x_N}\right]\left[\overline{-\sum_{\substack{|K|=q+1\\ j=0\ldots N}}\epsilon_{jI}^K\frac{\partial\psi_K}{\partial n}\frac{\partial\rho}{\partial x_j}}\right]\Bigg|_{x_N=f(x')}.\end{aligned}$$

Let $\epsilon \to 0$. Then

$$\frac{\partial f}{\partial x_k}\left[\epsilon^{3/4}\frac{\partial \chi}{\partial x_k}\right] \to 0 \quad \text{and} \quad \epsilon^{3/4}\frac{\partial \chi}{\partial x_N} \to 0.$$

Also, the hypothesis

$$\sum_{\substack{|K|=q+1\\ j=0\ldots N}} \epsilon^K_{jI}\frac{\partial \psi_K}{\partial n}\frac{\partial \rho}{\partial x_j} \neq 0$$

on $b\Omega$, together with the conditions on $\chi$, gives us as $\epsilon \to 0$ that

$$\int_{\mathbb{R}^N} \frac{\partial \phi_{I\epsilon}}{\partial n}\left[\overline{-\sum_{\substack{|K|=q+1\\ j=0\ldots N}} \epsilon^K_{jI}\frac{\partial \psi_K}{\partial n}\frac{\partial \rho}{\partial x_j}}\right]\bigg|_{x_N=f(x')} \sim \frac{1}{\epsilon^{1/4}}.$$

Next, we prove (i). Now,

$$\|\phi_{I\epsilon}\|_{W^1(\Omega)} = \int_\Omega |\phi_{I\epsilon}|^2\,dV + \sum_{k=0}^{N-1}\int_\Omega \left|\frac{\partial \phi_{I\epsilon}}{\partial x_k}\right|^2 dV + \int_\Omega \left|\frac{\partial \phi_{I\epsilon}}{\partial x_N}\right|^2 dV,$$

and by computation, we see that the first term is bounded by a constant.

Also,

$$\begin{aligned}\frac{\partial \phi_{I\epsilon}}{\partial x_k}(x', x_N) &= \frac{3}{4}\left(-\frac{\partial f}{\partial x_k}\right)\left(x_N - f(x') + \epsilon\right)^{-1/4}\chi(x', x_N) + \left(x_N - f(x') + \epsilon\right)^{3/4}\frac{\partial \chi}{\partial x_k}\\ &= \frac{3}{4}\left(-\frac{\partial f}{\partial x_k}\right)\left(x_N - f(x') + \epsilon\right)^{-1/4}\chi(x', x_N) + \Phi_\epsilon,\end{aligned}$$

where $\Phi_\epsilon \in C_0^\infty(\Omega)$ and is uniformly bounded in $\epsilon$. By computation, we see that the last two terms are also bounded by a constant.

So, (i) is established, and we have the required contradiction. The proposition is also proved. □

To give an expression for $d^*$, we note that (for $\psi \in \operatorname{dom} d^*$)

$$\langle d\phi, \psi\rangle_1 = \langle \phi, d'\psi\rangle_1 + \sum_{|I|=q}\int_{b\Omega}\phi_I\left(\overline{\sum_{\substack{|K|=q+1\\ j=0\ldots N}} \epsilon^K_{jI}\psi_K\frac{\partial \rho}{\partial x_j}}\right) + \sum_{\substack{|I|=q\\ k=0\ldots N}}\int_{b\Omega}\phi_I T_k^*\left[\overline{\sum_{\substack{|K|=q+1\\ j=0\ldots N}} \epsilon^K_{jI}D_k\psi_K\frac{\partial \rho}{\partial x_j}}\right].$$

We write $d^*\psi = d'\psi + \mathcal{K}\psi$ and we want to determine $\mathcal{K}\psi$.

**Proposition 2.** *Let $\psi \in \operatorname{dom} d^* \cap \Lambda^{q+1}(\overline{\Omega})$. Then $\mathcal{K}\psi$ is a q-form whose components satisfy*

$$\begin{cases} (\mathcal{K}\psi)_I - \Delta(\mathcal{K}\psi)_I = 0 & \text{in } \Omega,\\ \frac{\partial(\mathcal{K}\psi)_I}{\partial n} = \sum_{\substack{|K|=q+1\\ \alpha, j=0\ldots N}} \epsilon^K_{jI}T^*_\alpha\left[T_\alpha\psi_K\frac{\partial\rho}{\partial x_j}\right] + \left[\sum_{j=0}^N \epsilon^K_{jI}\psi_K\frac{\partial\rho}{\partial x_j}\right] & \text{on } b\Omega.\end{cases}$$

**Proof.** We have

$$\langle d\phi, \psi\rangle_1 = \langle \phi, d'\psi\rangle_1 + \langle \phi, \mathcal{K}\psi\rangle_1,$$

and from above, we see that for $\psi \in \operatorname{dom} d^* \cap \Lambda^{q+1}(\overline{\Omega})$,

$$\langle \phi, \mathcal{K}\psi\rangle_1 = \sum_{|I|=q}\int_{b\Omega}\phi_I\overline{\sum_{\substack{|K|=q+1\\ j=0\ldots N}} \epsilon^K_{jI}\psi_K\frac{\partial \rho}{\partial x_j}} + \sum_{\substack{|I|=q\\ k=0\ldots N}}\int_{b\Omega}\phi_I T_k^*\overline{\sum_{\substack{|K|=q+1\\ j=0\ldots N}} \epsilon^K_{jI}D_k\psi_K\frac{\partial \rho}{\partial x_j}}.$$

We also have

$$\begin{aligned}\langle\phi,\mathcal{K}\psi\rangle_1&=\sum_{|I|=q}\int_{\Omega}\phi_I\overline{(\mathcal{K}\psi)_I}+\sum_{\substack{|I|=q\\k=0\ldots N}}\int_{b\Omega}\phi_I\overline{D_k(\mathcal{K}\psi)_I\frac{\partial\rho}{\partial x_k}}-\sum_{\substack{|I|=q\\k=0\ldots N}}\int_{\Omega}\phi_I\overline{\Delta(\mathcal{K}\psi)_I}\\&=\sum_{|I|=q}\int_{\Omega}\phi_I\big[\overline{(\mathcal{K}\psi)_I-\Delta(\mathcal{K}\psi)_I}\big]+\sum_{|I|=q}\int_{b\Omega}\phi_I\frac{\overline{\partial(\mathcal{K}\psi)_I}}{\partial n}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{|I|=q}\int_{\Omega}\phi_I\big[\overline{(\mathcal{K}\psi)_I-\Delta(\mathcal{K}\psi)_I}\big]+\sum_{|I|=q}\int_{b\Omega}\phi_I\frac{\overline{\partial(\mathcal{K}\psi)_I}}{\partial n}\\&\quad=\sum_{|I|=q}\int_{b\Omega}\phi_I\overline{\sum_{\substack{|K|=q+1\\j=0\ldots N}}\epsilon^K_{jI}\psi_K\frac{\partial\rho}{\partial x_j}}+\sum_{\substack{|I|=q\\k=0\ldots N}}\int_{b\Omega}\phi_I T^*_k\overline{\sum_{\substack{|K|=q+1\\j=0\ldots N}}\epsilon^K_{jI}D_k\psi_K\frac{\partial\rho}{\partial x_j}},\end{aligned}$$

implying that

$$\begin{cases}(\mathcal{K}\psi)_I-\Delta(\mathcal{K}\psi)_I=0 & \text{in } \Omega,\\ \frac{\partial(\mathcal{K}\psi)_I}{\partial n}=\sum_{\substack{|K|=q+1\\j=0\ldots N}}\epsilon^K_{jI}\psi_K\frac{\partial\rho}{\partial x_j}+\sum_{k=0}^N T^*_k\sum_{\substack{|K|=q+1\\j=0\ldots N}}\epsilon^K_{jI}D_k\psi_K\frac{\partial\rho}{\partial x_j} & \text{on } b\Omega.\end{cases}$$

The result follows by writing $D_k\psi_K=T_k\psi_K+\frac{\partial\rho}{\partial x_k}\frac{\partial\psi_K}{\partial n}$ and noting that $\psi\in\operatorname{dom}d^*$. □

Now $d^*=d'+\mathcal{K}$ gives us

$$dd^*+d^*d=d(d'+\mathcal{K})+(d'+\mathcal{K})d=dd'+d\mathcal{K}+d'd+\mathcal{K}d=dd'+d'd+d\mathcal{K}+\mathcal{K}d\equiv-\Delta+G_\Omega,$$

where $G_\Omega=d\mathcal{K}+\mathcal{K}d$.

Using the results of Propositions 1 and 2, we can rewrite the boundary value problem (1) as

$$\begin{cases}(-\Delta+G_\Omega)\phi=\alpha & \text{in } \Omega,\\ \nabla_{N_1}\phi\lfloor N_1=0 & \text{on } b\Omega,\\ \nabla_{N_1}d\phi\lfloor N_1=0 & \text{on } b\Omega.\end{cases}\tag{2}$$

## 3. The diffeomorphism $\psi$

The boundary value problem (2) on $\mathbb{R}^{N+1}_+$ was studied in [2]. In order to make use of their results, we change the boundary value problem on $\Omega$ to a boundary value problem on $\mathbb{R}^{N+1}_+$ via a diffeomorphism.

Let $\psi:\Omega\to\mathbb{R}^{N+1}_+$ be a $C^\infty$ diffeomorphism such that $\psi(x_0,\ldots,x_N)=(s_0,\ldots,s_N)$, where $s=(s_0,\ldots,s_N)$ are the standard coordinates of $\mathbb{R}^{N+1}$. Then, for $j=0,\ldots,N$,

$$ds_j=\sum_{k=0}^N\frac{\partial s_j}{\partial x_k}\,dx_k.$$

Via the diffeomorphism $\psi^{-1}:\mathbb{R}^{N+1}_+\to\Omega$, where, for $j=0,\ldots,N$,

$$x_j=\psi_j^{-1}(s_0,\ldots,s_N),$$

our $q$-form $\phi$ is transformed into the $q$-form $\tilde\phi$ as follows:

$$\tilde\phi=\big(\psi^{-1}\big)^*(\phi)=\sum_{|I|=q}\phi_I\big(\psi^{-1}(s)\big)\sum_{\substack{|K|=q\\\pi\in\sum_N}}\operatorname{sgn}(\pi)\frac{\partial\psi^{-1}_{i_1}}{\partial s_{\pi(k_1)}}\cdots\frac{\partial\psi^{-1}_{i_q}}{\partial s_{\pi(k_q)}}\,ds_{k_1}\wedge\cdots\wedge ds_{k_q}\equiv\sum_{|K|=q}\tilde\phi_K\,ds^K,$$

where we have used

$$\big(\psi^{-1}\big)^*\big(dx^I\big)=\big(\psi^{-1}\big)^*(dx_{i_1})\wedge\cdots\wedge\big(\psi^{-1}\big)^*(dx_{i_q})$$

and

$$\left(\psi^{-1}\right)^*(dx_j)=\sum_{k=0}^{N}\frac{\partial\psi_j^{-1}}{\partial s_k}\,ds_k,$$

and where $\sum_N$ is the set of permutations of $\{0,1,\dots,N\}$ and $\operatorname{sgn}(\pi)$ is the signature of the permutation $\pi$.

In these coordinates, we have

$$d\tilde{\phi}=\sum_{|I|=q}\sum_{\substack{|K|=q+1\\ j=0\dots N}}\left(\epsilon^K_{jI}\frac{\partial\tilde{\phi}_I}{\partial s_j}\right)ds^K$$

and

$$d'\tilde{\phi}=-\sum_{|I|=q}\left(\sum_{\substack{|J|=q-1\\ j=0\dots N}}\epsilon^I_{jJ}D_j\tilde{\phi}_I\right)ds^J.$$

Let $N_1=\sum_{j=0}^N\frac{\partial\rho}{\partial x_j}\frac{\partial}{\partial x_j}$ be the normal vector field on $\Omega$, and $N_2=-\frac{\partial}{\partial s_N}$ be the normal vector field on $\mathbb{R}^{N+1}_+$. Then we have

**Proposition 3.** *There exists a $C^\infty$ function $\psi:\Omega\to\mathbb{R}^{N+1}_+$ such that $\psi_*(N_1)=N_2$ on the curve $x_N=f(x')$, where $\psi_*(N_1)$ is the push forward of $N_1$ under $\psi$. Also, $\psi$ is a diffeomorphism in an $\epsilon$-neighborhood of $b\Omega$.*

**Proof.**

We want to find a diffeomorphism $\psi$ such that

$$\psi_*(N_1)(s_0,\dots,s_N)=N_2(s_0,\dots,s_N).$$

By definition,

$$\psi_*(N_1)(s_0,\dots,s_N)=\begin{bmatrix}\frac{\partial\psi_0}{\partial x_0}&\frac{\partial\psi_0}{\partial x_1}&\cdots&\frac{\partial\psi_0}{\partial x_N}\\ \frac{\partial\psi_1}{\partial x_0}&\frac{\partial\psi_1}{\partial x_1}&\cdots&\frac{\partial\psi_1}{\partial x_N}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{\partial\psi_N}{\partial x_0}&\frac{\partial\psi_N}{\partial x_1}&\cdots&\frac{\partial\psi_N}{\partial x_N}\end{bmatrix}\begin{bmatrix}\frac{\partial\rho}{\partial x_0}\\ \frac{\partial\rho}{\partial x_1}\\ \vdots\\ \frac{\partial\rho}{\partial x_N}\end{bmatrix}$$

where $\rho(x',x_N)=f(x')-x_N$. So, we see that

$$\psi_*(N_1)(s_0,\dots,s_N)=\sum_{j=0}^N\sum_{k=0}^N\left[\frac{\partial\rho}{\partial x_k}\frac{\partial\psi_j}{\partial x_k}\right]ds_j.$$

So, we need to find $\psi$ such that, for $j=0,\dots,N-1$,

$$\sum_{k=0}^N\frac{\partial\psi_j}{\partial x_k}\frac{\partial\rho}{\partial x_k}=0 \tag{$A_j$}$$

and

$$\sum_{k=0}^N\frac{\partial\psi_N}{\partial x_k}\frac{\partial\rho}{\partial x_k}=-1 \tag{$A_N$}$$

on $x_N=f(x')$.

For $j=0,\dots,N$, let $b_j$ be $C^\infty(\mathbb{R}^{N+1})$ functions. Consider

$$\begin{aligned}\psi(x',x_N)=\bigl(&x_0+b_0(x',x_N)\bigl(x_N-f(x')\bigr),\dots,x_j+b_j(x',x_N)\bigl(x_N-f(x')\bigr),\dots,\\ &x_{N-1}+b_{N-1}(x',x_N)\bigl(x_N-f(x')\bigr),\bigl(x_N-f(x')\bigr)+b_N(x',x_N)\bigl(x_N-f(x')\bigr)\bigr).\end{aligned}$$

We want to determine $b_j$ so that $\psi_*(N_1) = N_2$ on the curve $x_N = f(x')$. With this $\psi$ (on $x_N = f(x')$), $(A_j)$ and $(A_N)$ become for $j = 0, \dots, N-1$,

$$\left(1 - b_j\left(\frac{\partial f}{\partial x_j}\right)\right)\left(\frac{\partial f}{\partial x_j}\right) + \sum_{\substack{k=0\dots N\\ k\neq j}}\left(-b_j\frac{\partial f}{\partial x_k}\right)\left(\frac{\partial f}{\partial x_k}\right) + b_j = 0 \tag{$A_j'$}$$

and

$$\sum_{k=0}^{N-1}\left(-\frac{\partial f}{\partial x_k} + b_N\left(-\frac{\partial f}{\partial x_k}\right)\right)\left(\frac{\partial f}{\partial x_k}\right) + (1+b_N)(-1) = -1. \tag{$A_N'$}$$

So, we require $b_j(x', f(x'))$ to be smooth functions on the curve $x_N = f(x')$ satisfying

$$b_j\big(x', f(x')\big) = \frac{\frac{\partial f}{\partial x_j}}{\sum_{k=0}^{N}\big(\frac{\partial f}{\partial x_k}\big)^2 + 1} \tag{$A_j''$}$$

and

$$b_N\big(x', f(x')\big) = \frac{-\sum_{k=0}^{N-1}\big(\frac{\partial f}{\partial x_k}\big)^2}{\sum_{k=0}^{N}\big(\frac{\partial f}{\partial x_k}\big)^2 + 1}. \tag{$A_N''$}$$

We wish to extend $b_j(x', f(x'))$ to all of $\mathbb{R}^{N+1}$. Let $\epsilon$ be small enough so that the lines normal to $x_N = f(x')$ do not intersect in an $\epsilon$-neighborhood of $x_N = f(x')$. Let $(\tilde{x}_0, \dots, \tilde{x}_N)$ be a point in this $\epsilon$-neighborhood. Extend $b_j(x', x_N)$ to $\tilde{b}_j(\tilde{x}_0, \dots, \tilde{x}_N)$ by letting $\tilde{b}_j(\tilde{x}_0, \dots, \tilde{x}_N) = b_j(x', f(x'))$, where $x'$ is the point on the curve $x_N = f(x')$ such that the line containing $(\tilde{x}_0, \dots, \tilde{x}_N)$ and $(x', f(x'))$ is normal to $x_N = f(x')$ at $x'$.

Next, let $\chi_{b_j}(x_0, \dots, x_N)$ be a $C^\infty$ cut-off function such that

$$\chi_{b_j}(x', x_N) = \begin{cases} 1 & \text{in } \frac{\epsilon}{2}\text{-nd of boundary,}\\ 0 & \text{outside } \epsilon\text{-nd of boundary.}\end{cases}$$

Then, we extend $\tilde{b}_j(\tilde{x}_0, \dots, \tilde{x}_N)$ to a function $B_j(x', x_N)$ in all of $\mathbb{R}^{N+1}$ by setting $B_j(x', x_N) = \chi_{b_j}(x', x_N) \times \tilde{b}_j(\tilde{x}_0, \dots, \tilde{x}_N)$.

Then, the diffeomorphism that we want is

$$\begin{aligned}\psi(x', x_N) = \big(&x_0 + B_1(x', x_N)\big(x_N - f(x')\big), \dots, x_j + B_j(x', x_N)\big(x_N - f(x')\big), \dots,\\ &x_{N-1} + B_{N-1}(x', x_N)\big(x_N - f(x')\big), \big(x_N - f(x')\big) + B_N(x', x_N)\big(x_N - f(x')\big)\big),\end{aligned}$$

since $B_j(x', f(x')) = b_j(x', f(x'))$. $\square$

## 4. Regularity

Let $\{O_p\}_{p\in\mathbb{N}}$ be an open covering of $\overline{\Omega} = \{(x', x_N) \in \mathbb{R}^{N+1} \mid x_N \geqslant f(x')\}$ with valence $k$ and satisfying the following two conditions: (1) If $O_p \cap b\Omega \neq \emptyset$, then $O_p$ is contained in the $\epsilon$-neighborhood of $b\Omega$ given in Proposition 3. (2) For each $p$, there is a positive integer $k_p$ such that for all $x \in O_p$, $k_p - \epsilon_0 < |x| < k_p + 1$, for some $0 < \epsilon_0 < 1$.

Let $\{\eta_p\}$ be a partition of unity subordinate to $\{O_p\}$ such that $(\frac{\partial}{\partial n})\eta_p = 0$ on a tubular neighborhood of $b\Omega$. Let $\phi$ be a solution to the boundary value problem (2).

Now, the open sets $O_p$ for fixed $p$ such that $O_p \cap b\Omega = \emptyset$ are interior to $\Omega$, and interior regularity follows from [2]. We concentrate on those $O_p$ such that $O_p \cap b\Omega \neq \emptyset$.

We could obtain a boundary regularity result by using a diffeomorphism from $\Omega$ to $\mathbb{R}^{N+1}_+$; however, the constant will depend on the particular diffeomorphism. In order to obtain a boundary regularity result in which the constant does not depend on any particular diffeomorphism, we introduce weight function spaces.

We define the *weighted norm* of a $q$-form $\phi$ with weight $w_s(x)$ to be

$$\|\phi\|^2_{s,w_s} = \sum_{|I|=q}\sum_{|\alpha|\leqslant s}\left\|\left(\frac{\partial^\alpha}{\partial x^\alpha}\phi_I\right)w_s\right\|_0^2.$$

Let $U$ be the $\epsilon$-neighborhood of $b\Omega$ given in Proposition 3. Let $\psi : U \to V \subseteq \mathbb{R}^{N+1}$ be a $C^\infty$ diffeomorphism such $\psi_*(N_1) = N_2$ as in Proposition 3. We note that $V$ is a neighborhood of $b\mathbb{R}^{N+1}_+$. Let $\psi_p \equiv \psi|_{O_p}$. We assume that the derivatives of $\psi_p$ and $\psi_p^{-1}$ are bounded; i.e. there exist $C^\infty$ functions $g_{\alpha,\beta,\ell}$ and $h_{\alpha,\beta,\ell}$ such that

$$\left|\left(\frac{\partial^{|\beta|}(\psi_p)_\ell}{\partial x^\beta}\right)^\alpha\right| \leqslant g_{\alpha,\beta,\ell}(k_p),$$

for $x \in O_p$ and

$$\left|\left(\frac{\partial^{|\beta|}(\psi_p^{-1})_\ell}{\partial t^\beta}\right)^\alpha\right| \leqslant h_{\alpha,\beta,\ell}(k_p),$$

for $t \in \psi_p^{-1}(O_p)$, and where

$$\left(\frac{\partial^{|\beta|}(\psi_p)_\ell}{\partial x^\beta}\right)^\alpha \equiv \left(\frac{\partial^{\beta_1}(\psi_p)_\ell}{\partial x^{\beta_1}}\right)^{\alpha_1}\left(\frac{\partial^{\beta_2}(\psi_p)_\ell}{\partial x^{\beta_2}}\right)^{\alpha_2}\cdots\left(\frac{\partial^{\beta_m}(\psi_p)_\ell}{\partial x^{\beta_m}}\right)^{\alpha_m}$$

and

$$\left(\frac{\partial^{|\beta|}(\psi_p^{-1})_\ell}{\partial t^\beta}\right)^\alpha \equiv \left(\frac{\partial^{\beta_1}(\psi_p^{-1})_\ell}{\partial t^{\beta_1}}\right)^{\alpha_1}\left(\frac{\partial^{\beta_2}(\psi_p^{-1})_\ell}{\partial t^{\beta_2}}\right)^{\alpha_2}\cdots\left(\frac{\partial^{\beta_m}(\psi_p^{-1})_\ell}{\partial t^{\beta_m}}\right)^{\alpha_m}.$$

We note that $g_{\alpha,\beta,\ell}(k_p)$ and $h_{\alpha,\beta,\ell}(k_p)$ are reciprocals of each other.

In particular,

$$\left|\mathrm{Jac}(\psi_p)\right| \leqslant C \max_{\substack{|\alpha|=N+1\\|\beta|=1,\,\ell=0\ldots N}} \left[g_{\alpha,\beta,\ell}(k_p)\right]$$

and

$$\left|\mathrm{Jac}\left(\psi_p^{-1}\right)\right| \leqslant C \max_{\substack{|\alpha|=N+1\\p|\beta|=1,\,\ell=0\ldots N}} \left[h_{\alpha,\beta,\ell}(k_p)\right].$$

We look at some inequalities.

**Lemma 1.** *The $q$-form $\phi$ satisfies*

$$\|\phi\|_{s,w_s} \leqslant C\|\tilde{\phi}\|_s,$$

*where the weight $w_s = w_s(x)$ is*

$$w_s(x) \equiv \frac{1}{\max_{q\leqslant|\alpha|\leqslant q+s,\,|\beta|\leqslant s+1,\ell}[g_{\alpha,\beta,\ell}(|x|)]} \times \frac{1}{(\max_{|\alpha|=N+1,\,|\beta|=1,\ell}[h_{\alpha,\beta,\ell}(|\psi(x)|)])^{1/2}}.$$

**Proof.**

$$\begin{aligned}
\|\phi\|^2_{s,w_s} &= \left\|(\psi)^*(\tilde{\phi})\right\|^2_{s,w_s} \\
&\leqslant \sum_{|I|=q}\sum_{|\ell|\leqslant s}\sum_j \int_U \eta_j(x)\left|\frac{\partial^{|\ell|}\tilde{\phi}_I}{\partial t^\ell}\big(\psi(x)\big)\right|^2 \frac{1}{\max_{\substack{|\alpha|=N+1\\|\beta|=1,\ell}}[h_{\alpha,\beta,\ell}(|\psi(k_j)|)]}\,dx_0\ldots dx_N \\
&\leqslant \sum_{|I|=q}\sum_{|\ell|\leqslant s}\sum_j \int_V \eta_j\big(\psi^{-1}(t)\big)\left|\frac{\partial^{|\ell|}\tilde{\phi}_I}{\partial t^\ell}(t)\right|^2 \frac{1}{\max_{\substack{|\alpha|=N+1\\|\beta|=1,\ell}}[h_{\alpha,\beta,\ell}(|\psi(k_j)|)]}\left|\mathrm{Jac}\,\psi^{-1}\right| dt_0\ldots dt_N \\
&\leqslant \sum_{|I|=q}\sum_{|\ell|\leqslant s}\int_V \left|\frac{\partial^{|\ell|}\tilde{\phi}_I}{\partial t^\ell}(t)\right|^2 dt_0\ldots dt_N = C\|\tilde{\phi}\|^2_s. \qquad \square
\end{aligned}$$

Now, we look at the opposite inequality.

**Lemma 2.** *The $q$-form $\phi$ satisfies*

$$\|\tilde{\phi}\|_s \leqslant C\|\phi\|_{s,\tilde{w}_s},$$

*where the weight $\tilde{w}_s = \tilde{w}_s(t)$ is*

$$\tilde{w}_s(t) \equiv \max_{\substack{q\leqslant|\alpha|\leqslant q+s \\ |\beta|\leqslant s+1,\ell}} \big[h_{\alpha,\beta,\ell}\big(|t|\big)\big] \cdot \Big(\max_{\substack{|\alpha|=N+1 \\ |\beta|=1,\ell}} \big[g_{\alpha,\beta,\ell}\big(\big|\psi^{-1}(t)\big|\big)\big]\Big)^{1/2}.$$

**Proof.** We proceed as above:

$$\begin{aligned}
\|\tilde{\phi}\|_s^2 &\leqslant C\sum_{|I|=q}\sum_{|m|\leqslant s}\sum_j \int_V \eta_j\big(\psi^{-1}(t)\big)\left|\frac{\partial^{|m|}}{\partial t^m}\big(\phi_I\big(\psi^{-1}(t)\big)\big)\right|^2 \max_{\substack{q\leqslant|\alpha|\leqslant q+s \\ |\beta|\leqslant s+1,\ell}} \big[h_{\alpha,\beta,\ell}(k_j)\big]^2\, dt_0\ldots dt_N \\
&\leqslant C\sum_{|I|=q}\sum_{|m|\leqslant s}\sum_j \int_U \eta_j(x)\left|\frac{\partial^{|m|}}{\partial x^m}\big(\phi_I(x)\big)\right|^2 \max_{\substack{q\leqslant|\alpha|\leqslant q+s \\ |\beta|\leqslant s+1,\ell}} \big[h_{\alpha,\beta,\ell}(k_j)\big]^2 |\operatorname{Jac}\psi|\, dx_0\ldots dx_N \\
&\leqslant C\sum_{|I|=q}\sum_{|m|\leqslant s} \int_U \left|\frac{\partial^{|m|}}{\partial x^m}\big(\phi_I(x)\big)\right|^2 dx_0\ldots dx_N \\
&\leqslant C\|\phi\|_s^2. \qquad \square
\end{aligned}$$

We have computed these inequalities with weighted norms with a view to proving a regularity result for this boundary value problem where the constant does not depend on any diffeomorphism transforming $\Omega$ to $\mathbb{R}_+^{N+1}$.

We use Lemmas 1 and 2 to prove our regularity result.

**Theorem 1.** *The $q$-form $\phi$ on a neighborhood of $b\Omega$ satisfies*

$$\|\phi\|_{s+2,w_{s+2}} \leqslant C\big(\|\alpha\|_{s,\tilde{w}_s} + \|\phi\|_{s+1,\tilde{w}_{s+1}}\big).$$

*The constant $C$ depends on $O_j$, $s$, and $\eta_j$.*

**Proof.** By Lemmas 1 and 2, and Theorem 11.3 from [2] we have:

$$\begin{aligned}
\|\phi\|_{s+2,w_{s+2}} &\leqslant C\|\tilde{\phi}\|_{s+2} \leqslant C\sum_j \|\tilde{\eta}_j\tilde{\phi}\|_{s+2} \leqslant C\sum_j\big(\big\|\tilde{\eta}_{j1}(-\Delta + G_{\mathbb{R}_+^{N+1}})(\tilde{\eta}_j\tilde{\phi})\big\|_s + \|\tilde{\eta}_j\tilde{\phi}\|_{s+1}\big) \\
&\leqslant C\sum_j\big(\big\|\tilde{\eta}_j(\widetilde{-\Delta + G_\Omega})(\tilde{\phi})\big\|_s + \big\|-\tilde{\eta}_j\widetilde{\Delta\phi} + \Delta(\tilde{\eta}_j\tilde{\phi})\big\|_s \\
&\qquad + \big\|\tilde{\eta}_{j1}G_{\mathbb{R}_+^{N+1}}(\tilde{\eta}_j\tilde{\phi}) - \tilde{\eta}_{j1}\tilde{\eta}_j\widetilde{G_\Omega\phi}\big\|_s + \|\tilde{\eta}_j\tilde{\phi}\|_{s+1}\big) \\
&\leqslant C\sum_j\big(\|\tilde{\eta}_j\tilde{\alpha}\|_s + \big\|\tilde{\eta}_{j1}G_{\mathbb{R}_+^{N+1}}(\tilde{\eta}_j\tilde{\phi}) - \tilde{\eta}_{j1}\tilde{\eta}_j\widetilde{G_\Omega\phi}\big\|_s + \|\tilde{\eta}_j\tilde{\phi}\|_{s+1}\big) \\
&\leqslant C\sum_j\big(\|\tilde{\eta}_j\tilde{\alpha}\|_s + \epsilon\|\tilde{\eta}_j\tilde{\phi}\|_{s+2} + C_\epsilon\|\tilde{\eta}_{j1}\tilde{\phi}\|_s + \|\tilde{\eta}_j\tilde{\phi}\|_{s+1}\big) \leqslant C\sum_j\big(\|\tilde{\eta}_j\tilde{\alpha}\|_s + \|\tilde{\eta}_{j1}\tilde{\phi}\|_{s+1}\big) \\
&\leqslant C\sum_j\big(\|\eta_j\alpha\|_{s,\tilde{w}_s} + \|\eta_{j1}\phi\|_{s+1,\tilde{w}_{s+1}}\big) \leqslant C\sum_j\big(\|\eta_j\alpha\|_{s,\tilde{w}_s} + \|\phi\|_{s+1,\tilde{w}_{s+1}}\big) \\
&\leqslant C\big(\|\alpha\|_{s,\tilde{w}_s} + \|\phi\|_{s+1,\tilde{w}_{s+1}}\big). \qquad \square
\end{aligned}$$

Finally, we have one more inequality which we will use to prove the existence statement.

**Proposition 4.** *Fix $p$. Let $k_p$ be such that $k_p - \epsilon_0 < |x| < k_p + 1$ for all $x \in O_p$. Then there exist constants $c_0$ and $c_1$ such that*

$$c_0\|\phi\|_1 \leqslant \|\phi\|_{1,w_1} \leqslant c_1\|\phi\|_1.$$

**Proof.** We have:

$$\|\phi\|_{1,w_1}^2 = \sum_{|\alpha|\leqslant 1} \left\| \left( \frac{\partial^\alpha}{\partial x^\alpha} \phi \right) w_1 \right\|_0^2 = \sum_{|\alpha|\leqslant 1} \int_{O_p} \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi w_1(x) \right|^2 dV.$$

The weight, on a fixed $O_p$, has a maximum value, say $c_1$, and a minimum value, say $c_0$. Thus

$$\|\phi\|_{1,w_1}^2 \leqslant c_1 \sum_{|\alpha|\leqslant 1} \int_{O_p} \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi \right|^2 dV = c_1 \|\phi\|_1^2$$

and

$$\|\phi\|_{1,w_1}^2 \geqslant c_0 \sum_{|\alpha|\leqslant 1} \int_{O_p} \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi \right|^2 dV = c_0 \|\phi\|_1^2. \qquad \square$$

## 5. Existence

We want to study the existence of a solution to the boundary value problem (2) for $\alpha \in \Lambda^1(\overline{\Omega})$ from an abstract Hilbert space point of view. We follow the same method as in [2].

Let $W_{q,w_1}^1$ be $q$-forms such that the weighted norm $\|\cdot\|_{1,w_1}$ is finite. Let $\Lambda_{w_1}^q(\overline{\Omega})$ be the set of $q$-forms with smooth coefficients in $W_{q,w_1}^1$. Let

$$\mathcal{D} \equiv \{\phi \in \Lambda_{w_1}^q(\overline{\Omega}):\ \phi, d\phi \in \operatorname{dom} d^*\}.$$

For $\phi, \psi \in \mathcal{D}$, we define the bilinear forms

$$Q(\phi,\psi) = \langle d\phi, d\psi\rangle_1 + \langle d^*\phi, d^*\psi\rangle_1 + \langle \phi,\psi\rangle_1$$

and

$$Q_1(\phi,\psi) = \langle d\phi, d\psi\rangle_{1,w_1} + \langle d^*\phi, d^*\psi\rangle_{1,w_1} + \langle \phi,\psi\rangle_{1,w_1}.$$

We first claim that $\mathcal{D}$ is dense in $W_{q,w_1}^1$. Let $\phi$ be any $q$-form in $\Lambda_{w_1}^q$ with coefficients smooth up to the boundary. It suffices to find a $\psi \in \Lambda_{w_1}^q(\overline{\Omega})$ of small norm such that $\phi + \psi \in \mathcal{D}$. We use local coordinates in a tubular neighborhood of $b\Omega$, so that $x = (x', x_N)$ with $x' \in b\Omega$ and $x_N = \operatorname{dist}(x_N, b\Omega)$ parametrizing the normal direction. Set

$$\psi_1(x') = \nabla_{\vec{n}}(\phi \lfloor \vec{n})|_{b\Omega}$$

and

$$\psi_2(x') = \nabla_{\vec{n}}(d\phi \lfloor \vec{n})|_{b\Omega}\, d\psi_1(x').$$

Set

$$\psi(x) = \left( -x_N\, dx_N \wedge \psi_1(x') - \frac{1}{2} x_N^2 \psi_2(x') \right) \chi(x_N),$$

where $\chi \in C_0^\infty[-2\epsilon, 2\epsilon]$, $\chi = 1$ in $[-\epsilon,\epsilon]$ and $\|\chi\|_1 \leqslant C\epsilon^{-1/2}$. Then, as in [2, page 65], $\|\psi\|_1 < C\epsilon^{1/2}$ and $\phi + \psi \in \mathcal{D}$.

Now, let $\tilde{\mathcal{D}}$ be the closure of $\mathcal{D}$ in the topology induced by $Q_1$. We wish to check that $\tilde{\mathcal{D}}$ is still contained in $W_{q,w_1}^1$. Let $\{\phi_n\}$ be a Cauchy sequence in $\mathcal{D}$ in the $Q_1$ topology. By the definition of $Q_1$, we see that $\{\phi_n\}$ and $\{d\phi_n\}$ are Cauchy sequences in $W_{q,w_1}^1$. Let $\phi$ be the $W_{q,w_1}^1$ limit of $\{\phi_n\}$. We see that $d$ is closed in $W_{q,w_1}^1$, and since adjoints are always closed, it follows that $d^*$ is also closed. So we have

$$d\phi_n \to d\phi.$$

If $\phi = 0$ in $W_{q,w_1}^1$, then

$$Q_1(\phi,\phi) = \lim_n Q_1(\phi_n,\phi_n) = \lim_n \bigl[\langle d\phi_n, d\phi_n\rangle_{1,w_1} + \langle d^*\phi_n, d^*\phi_n\rangle_{1,w_1} + \langle \phi_n,\phi_n\rangle_{1,w_1}\bigr]$$
$$= \langle d\phi, d\phi\rangle_{1,w_1} + \langle d^*\phi, d^*\phi\rangle_{1,w_1} + \langle \phi,\phi\rangle_{1,w_1} = 0.$$

Thus, $\phi_n \to 0$ in the $Q_1$ topology. So, the inclusion $\mathcal{D} \hookrightarrow W^1_{q,w_1}$ extends to $\tilde{\mathcal{D}} \hookrightarrow W^1_{q,w_1}$. So, $\tilde{\mathcal{D}}$ is a subspace of $W^1_{q,w_1}$.

In order to continue our analysis, we apply the Friedrichs extension theorem to show that there exists a canonical self adjoint operator

$$T : W^1_{q,w_1} \to \tilde{\mathcal{D}}$$

which is bounded in the $W^1_{q,w_1}$ topology, is injective, and such that

$$Q_1(T\phi,\psi) = \langle \phi,\psi\rangle_{1,w_1}.$$

Now, if we set $F = T^{-1}$, then

$$Q_1(\phi,\psi) = \langle F\phi,\psi\rangle_{1,w_1} \quad \forall \phi \in \operatorname{dom} F,\ \psi \in \tilde{\mathcal{D}}.$$

Notice that $F = d^*d + dd^* + I$ when restricted to $\mathcal{D}$:

$$Q_1(\phi,\psi) = \langle d\phi, d\psi\rangle_{1,w_1} + \langle d^*\phi, d^*\psi\rangle_{1,w_1} + \langle \phi,\psi\rangle_{1,w_1} = \langle d^*d\phi,\psi\rangle_{1,w_1} + \langle dd^*\phi,\psi\rangle_{1,w_1} + \langle \phi,\psi\rangle_{1,w_1}$$
$$= \bigl\langle (d^*d + dd^* + I)\phi,\psi\bigr\rangle_{1,w_1}.$$

The following proposition comes from [1].

**Proposition 5.** *On $\mathcal{D}$, we have $T \equiv (I + dd^*)^{-1} + (I + d^*d)^{-1} - I = F^{-1}$.*

Let us assume that the $Q_1$-unit ball is compact in $W^1_{q,w_1}$. Then, it follows that $T$ is a compact operator [1]. Notice that if $T\alpha \in \mathcal{D}$, then $T\alpha$ is the unique solution of the boundary value problem

$$\begin{cases} (d^*d + dd^*)\phi + \phi = \alpha & \text{in } \Omega, \\ \phi, d\phi \in \operatorname{dom} d^* \end{cases}$$

for $\alpha \in W^1$, because if $T\alpha \in \mathcal{D}$, then $d(T\alpha)$ and $T\alpha \in \operatorname{dom} d^*$ and

$$(d^*d + dd^*)(T\alpha) + (T\alpha) = (d^*d + dd^* + I)(T\alpha) = F(T\alpha) = \alpha.$$

Now, we wish to determine what conditions are needed to solve

$$\begin{cases} (d^*d + dd^*)\phi = \alpha & \text{in } \Omega, \\ \phi, d\phi \in \operatorname{dom} d^*. \end{cases}$$

Let $Q_0$ be the bilinear form on $\tilde{\mathcal{D}}$ defined by

$$Q_0(\phi,\psi) = \langle d\phi, d\psi\rangle_{1,w_1} + \langle d^*\phi, d^*\psi\rangle_{1,w_1}.$$

So $\phi$ is a solution to $(*)$ exactly when $\phi \in \mathcal{D}$ and

$$Q_0(\phi,\psi) = \langle \alpha,\psi\rangle_{1,w_1} \quad \forall \psi \in \tilde{\mathcal{D}}.$$

But notice that this equality holds if and only if

$$Q_1(\phi,\psi) = Q_0(\phi,\psi) + \langle \phi,\psi\rangle_{1,w_1} = \langle \alpha + \phi,\psi\rangle_{1,w_1}.$$

By the Friedrichs extension theorem, we need only solve the equation

$$(F - I)\phi = \alpha \tag{$**$}$$

with $\phi \in \operatorname{dom} F$.

Thus setting $\theta = F\phi$, we have the equation

$$\theta - T\theta = \alpha.$$

By the theory of compact operators, the above equation has a solution $\theta$ for all $\alpha$ orthogonal to the finite dimensional subspace $H \equiv \ker(I - T)$. The following theorem is from [4].

**Theorem 2.** *If $T$ is a compact operator on a Hilbert space $H$, then $Tx = \lambda x + y$ has a solution if and only if $y \perp z$ for all $z$ with $T^* z = \bar{\lambda} z$.*

We want to solve the equation

$$\theta - T\theta = \alpha.$$

By the above theorem, $T\theta = \theta - \alpha$ has a solution if and only if $-\alpha \perp \beta$ for all $\beta$ with $T^*\beta = \beta$. Since $T$ is self adjoint, we have $T\theta = \theta - \alpha$ has a solution if and only if

$$-\alpha \perp \beta \quad \forall \beta \text{ with } T\beta = \beta \text{ i.e. } \forall \beta \in \ker(T - I).$$

So, $T\theta = \theta - \alpha$ has a solution $\theta$ if and only if $\alpha$ is orthogonal to $\ker(T - I)$. We note that $\ker(I - T)$ is exactly $\ker(F - I)$.

Hence for $\alpha$ orthogonal to $\ker(I - T)$, we have that $\phi = T\theta$ is the solution of equation $(**)$. Notice also that if $\phi \in \mathcal{D}$, then equation $(**)$ reduces to $(*)$. Moreover, $Z \equiv (F - I)(\mathcal{D})$ is a dense subspace of $H^\perp \subset W^1$, i.e. $Z^\perp = H$.

Thus, once we have proven the assertion about $Q_1$, we will have proved the following theorem:

**Theorem 3** *(Existence). The boundary value problem*

$$\begin{cases} (dd^* + d^*d)\phi = \alpha & \text{in } \Omega, \\ \phi \in \operatorname{dom} d^*, \\ d\phi \in \operatorname{dom} d^* \end{cases}$$

*has a finite dimensional kernel $H$ and finite dimensional cokernel. The problem has a solution $\phi \in \mathcal{D}$ for $\alpha \in Z \subset H$.*

Now, we need to prove the claim that the $Q_1$-unit ball is compact in $W^1_{q,w_1}$. We start with a lemma.

**Lemma 3.** *The $Q_1$-unit ball is compact in $W^1_{q,w_1}$ on the bounded set $O_p$ (as defined in Section 4).*

**Proof.** For $\phi \in \mathcal{D}$, we restrict $\phi$ to $O_p$. Then, we define $Q_{1p}(\phi) \equiv \|d\phi\|^2_{1,w_1} + \|d^*\phi\|^2_{1,w_1} + \|\phi\|^2_{1,w_1}$ and $Q(\phi) \equiv \|d\phi\|^2_1 + \|d^*\phi\|^2_1 + \|\phi\|^2_1$ on $O_p$. The $Q_{1p}$-unit ball is defined to be $\{Q_{1p}(\phi)\colon \|\phi\|_{1,w_1} \leqslant 1\}$, and the $Q$-unit ball is likewise defined to be $\{Q(\phi)\colon \|\phi\|_1 \leqslant 1\}$.

Let $\{\phi_k\}$ be a sequence of $q$-forms in $\mathcal{D}$ restricted to $O_p$ such that $\|\phi_k\|_1 \leqslant 1/c_1$, where $c_1$ is the constant from Proposition 4. We note that since $\|\phi\|_{1,w_1} \leqslant c_1\|\phi\|_1$, we have that $\|\phi_k\|_{1,w_1} \leqslant 1$ also. We need to show that there exists a convergent subsequence of $Q_{1p}(\phi_k)$.

Now, since the $Q$-unit ball is compact on bounded sets [2], there is a subsequence $\{Q(\phi_{k_\ell})\}$ which converges in $W^1_q$ to $Q(\phi)$ such that $\|\phi\|_1 \leqslant 1$. So, $\{Q(\phi_{k_\ell})\}$ is a Cauchy sequence. Let $\epsilon > 0$. Choose $M$ so large that for $m, n > M$, we have $Q(\phi_{k_m}) - Q(\phi_{k_n}) < \epsilon$. Then, we have

$$\begin{aligned} &\left|Q_{1p}(\phi_{k_m}) - Q_{1p}(\phi_{k_n})\right| \\ &\quad = \left|\|d\phi_{k_m}\|^2_{1,w_1} - \|d\phi_{k_n}\|^2_{1,w_1} + \|d^*\phi_{k_m}\|^2_{1,w_1} - \|d^*\phi_{k_n}\|^2_{1,w_1} + \|\phi_{k_m}\|^2_{1,w_1} - \|\phi_{k_n}\|^2_{1,w_1}\right| \\ &\quad \leqslant \left|c_1\|d\phi_{k_m}\|^2_1 - c_0\|d\phi_{k_n}\|^2_1 + c_1\|d^*\phi_{k_m}\|^2_1 - c_0\|d^*\phi_{k_n}\|^2_1 + c_1\|\phi_{k_m}\|^2_1 - c_0\|\phi_{k_n}\|^2_1\right| \\ &\quad \leqslant \left|c_1 Q(\phi_{k_m}) - c_0 Q(\phi_{k_n})\right| \leqslant c\left|Q(\phi_{k_m}) - Q(\phi_{k_n})\right| < c\epsilon. \end{aligned}$$

So, $\{Q_{1p}(\phi_{k_m})\}$ is a convergent sequence, and the result follows. □

Now, we use this lemma to show the $Q_1$-unit ball is compact in $W^1_{q,w_1}$.

**Theorem 4.** *The $Q_1$-unit ball is compact in $W^1_{q,w_1}$.*

**Proof.** Let $\{\phi_k\}$ be a sequence of $q$-forms in $\mathcal{D}$ such that $\|\phi_k\|_1 \leqslant 1/c_1$, where $c_1$ is the constant from Proposition 4. We note that since $\|\phi\|_{1,w_1} \leqslant c_1\|\phi\|_1$, we have that $\|\phi_k\|_{1,w_1} \leqslant 1$ also. We need to show that there exists a convergent subsequence of $Q_1(\phi_k)$.

Renumber the open cover $\{O_p\}_{p\in\mathbb{N}}$ as $\{O_{p_1}, O_{p_2}, \dots, O_{p_n}, \dots\}$. By Lemma 3, when we restrict $\phi_k$ to $O_{j_1}$, then there is a subsequence of $\{Q_{1p_1}(\phi_k)\}$, say $\{Q_{1p_1}(\phi_{k_m})\}$, which converges to $\{Q_{1p_1}(\phi)\}$.

Next, consider the sequence $\{\phi_{k_m}\}$. When we restrict this sequence to $O_{p_2}$, then Lemma 3 tells us that there is a convergent subsequence of $\{Q_{1p_2}(\phi_{k_m})\}$, say $\{Q_{1p_2}(\phi_{k_{m_n}})\}$.

Continuing in this manner (and renumbering each time), we obtain a converging sequence $\{Q_{1p_\ell}(\phi_j)\} = \{Q_1(\phi_j)|_{O_{p_\ell}}\}$ in $W^1_{q,w_1}(\bigcup_{k\leqslant\ell} O_{p_k})$. We have

$$\begin{array}{llllll}
Q_1(\phi_1)|_{O_{p_1}} & Q_1(\phi_2)|_{O_{p_1}} & Q_1(\phi_3)|_{O_{p_1}} & \cdots & Q_1(\phi_{p_1})|_{O_{p_1}} \in W^1_{q,w_1}(O_{p_1}), \\
Q_1(\phi_1)|_{O_{p_2}} & Q_1(\phi_2)|_{O_{p_2}} & Q_1(\phi_3)|_{O_{p_2}} & \cdots & Q_1(\phi_{p_2})|_{O_{p_2}} \in W^1_{q,w_1}\big(\bigcup_{k=1}^2 O_{p_k}\big), \\
Q_1(\phi_1)|_{O_{p_3}} & Q_1(\phi_2)|_{O_{p_3}} & Q_1(\phi_3)|_{O_{p_3}} & \cdots & Q_1(\phi_{p_3})|_{O_{p_3}} \in W^1_{q,w_1}\big(\bigcup_{k=1}^3 O_{p_k}\big), \\
\vdots & \vdots & \vdots & \ddots &
\end{array}$$

Notice that if $\ell < n$, then $\{Q_1(\phi_j)|_{O_{p_n}}\}$ is a subsequence of $\{Q_1(\phi_j)|_{O_{p_\ell}}\}$. Thus, for any $j, n$ and $\ell$ (with $\ell < n$), we can find a $k$ so that $\{Q_1(\phi_j)|_{O_{p_n}}\} = \{Q_1(\phi_k)|_{O_{p_\ell}}\}$ on $W^1_{q,w_1}(\bigcup_{m=1}^n O_{p_m})$.

Let us consider the sequence $\{Q_1(\phi_n)|_{O_{p_n}}\}$. We will show that this sequence converges in $W^1_{q,w_1}(\Omega)$. Let $\epsilon > 0$.

First notice that there is a number $N$ such that for $j, k > N$ we have

$$\big|Q_1(\phi_j)|_{O_{p_n}} - Q_1(\phi_k)|_{O_{p_n}}\big| < \frac{\epsilon}{2}.$$

So, for $n, m, j, k > N$ with $m < n$ and $k$ chosen so that $\{Q_1(\phi_j)|_{O_{p_n}}\} = \{Q_1(\phi_k)|_{O_{p_m}}\}$ we have

$$\begin{aligned}
\big|Q_1(\phi_n)|_{O_{p_n}} - Q_1(\phi_m)|_{O_{p_m}}\big| &< \big|Q_1(\phi_n)|_{O_{p_n}} - Q_1(\phi_j)|_{O_{p_n}}\big| + \big|Q_1(\phi_j)|_{O_{p_n}} - Q_1(\phi_k)|_{O_{p_m}}\big| \\
&\quad + \big|Q_1(\phi_k)|_{O_{p_m}} - Q_1(\phi_m)|_{O_{p_m}}\big| \\
&< \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

So, the sequence converges, and the theorem follows. □

Thus we have shown that for $\alpha$ orthogonal to the kernel of the boundary value problem

$$\begin{cases}
(dd^* + d^*d)\phi = \alpha & \text{in } \Omega, \\
\phi \in \operatorname{dom} d^*, \\
d\phi \in \operatorname{dom} d^*
\end{cases}$$

there is a solution $\phi$ in $\mathcal{D}$.

## 6. Conclusion

We have been studying the boundary value problem on an unbounded domain $\Omega$ with smooth boundary obtained from the elliptic pseudo-differential equation $(dd^* + d^*d)\phi = \alpha$, where the adjoint $d^*$ is calculated in the $W^1$ topology. The domain of $d^*$ was determined, and an expression for $d^*$ was found. Regularity and existence results are given.

The author would like to continue to study the boundary value problem on unbounded domains in which the adjoint operator $d^*$ is calculated in the $W^s$ topology.

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